Strongly asymmetric soliton explosions

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We have observed, in numerical simulations, strongly asymmetric soliton explosions in dissipative systems modeled by the one-dimensional complex cubic-quintic Ginzburg-Landau equation. The explosions occur at one side of the soliton, in spite of the fact that the initial conditions and the equation itself are symmetric. From one explosion to the next, the side of the soliton where it occurs alternates, so that the left- and right-hand sides of the soliton explode successively. We give explanations for this effect based on a linear stability analysis of the unstable soliton. We also describe the transition from the stationary soliton into the exploding one as a three-stage process of soliton cooling.

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I. INTRODUCTION

Dissipative solitons are basic localized modes defining the dynamics of many systems far from equilibrium. Their properties are unique and quite different from those of solitons in conservative systems. Several models of dissipative systems have been studied, including those described by the complex Ginzburg-Landau (CGLE) and the complex Swift-Hohenberg equations. These equations apply to a wide range of dissipative phenomena in physics, such as binary fluid convection [1], electroconvection in nematic liquid crystals [2], patterns near electrodes in gas discharges [3], and oscillatory chemical reactions [4]. The cubic CGLE was derived for supercritical bifurcations. Its cubic-quintic extension serves as a generic equation describing systems near subcritical bifurcations [5,6]. This equation contains the basic terms describing the most important physical phenomena occurring in passively mode-locked lasers [7].

In contrast to conservative solitons, solitons in systems far from equilibrium are dynamical objects that have nontrivial internal energy flows. Since they are produced by a dissipative system, they depend strongly on an energy supply from an external source. In spite of being a stationary object, a dissipative soliton continuously redistributes energy between its parts. A soliton that periodically changes its shape in time (pulsating soliton [8]) can be considered as another object (limit cycle) in an infinite-dimensional phase space. These can also be stable or unstable. Stable pulsating solitons exist for an indefinite time, in the same way as stationary solutions. A pulsation that appears in the soliton profile is just the external revelation of its internal dynamics. Dissipative solitons can take a variety of shapes and also show a variety of periodic and chaotic changes. One of the most striking behaviors that we have observed numerically [8,9] and experimentally [10] is that of "explosions." These are soliton solutions that periodically suffer explosive instability, but return to the original shape after each explosion. The solution serves as an attractor in the sense that a wide variety of localized initial conditions converge to it.

In the first work on exploding (erupting) solitons [9], it was found that these localized objects exist over a wide range of the system parameters (see Fig. 5 of [9]). In fact, the range of parameters is, if not larger than, then at least comparable with the range where stable stationary solitons exist. This makes their study quite worthwhile, both theoretically and experimentally. This remarkable property should also make their observation a relatively easy task. Some explanations for the existence of exploding solitons and their unusual dynamics were presented in our recent work [11].

The existence of internal dynamics in the dissipative soliton creates a variety of new instabilities that do not exist in conservative systems. This fact makes the study of its stability more difficult than that in conservative systems. For stationary solitons, we can apply linear stability analysis, but, in the case of pulsating solitons, that is not possible. In the case of exploding solitons, there are two clearly distinguished regions: the laminar stage of propagation and the exploding stage. Therefore, in this particular case, we can use the linear stability analysis, at least partially, and apply it to the soliton in its laminar regime when the soliton is very close to being stationary. In this way, we can answer at least some of the questions related to its unusual behavior.

In the present work, we have found that soliton explosions can be strongly asymmetric; namely, each explosion happens predominantly at one side of the soliton rather than at both sides simultaneously. From one explosion to another, the side of the soliton where it occurs alternates, so that explosions occur at the left- (LHS) and right-hand sides (RHS) of the soliton successively. Taking into account that the soliton itself in the laminar regime of propagation is symmetric and that there is no asymmetry in the equation, these strongly asymmetric explosions need explanations.

These explanations can be obtained if we study the stability of the stationary solution that serves as a foundation for the exploding soliton. The set of linear eigenfunctions and the eigenvalues that we found turned out to be almost degenerate, and consisted of a symmetric and an antisymmetric mode, as should occur in a symmetric system. In addition, we have found that the eigenvalues for these two modes are slightly different. This causes them to compete, and the instability becomes highly asymmetric due to the nonlinearity.

II. THE MODEL

The cubic-quintic complex Ginzburg-Landau equation can be written

$$i\psi_z + \frac{D}{2}\psi_{tt} + |\psi|^2\psi + \nu|\psi|^4 = i\delta\psi + i\epsilon|\psi|^2\psi + i\beta\psi_{tt} + i\mu|\psi|^4\psi. \tag{1}$$

When used to describe passively mode-locked lasers, z is the cavity round-trip number, t is the retarded time, ψ is the normalized envelope of the field, D is the group velocity dispersion coefficient, with $D=\pm 1$, depending on whether the group velocity dispersion is anomalous or normal, respectively, δ is the linear gain-loss coefficient, $i\beta\psi_{tt}$ accounts for spectral filtering $(\beta>0)$), $\epsilon|\psi|^2\psi$ represents the nonlinear gain (which arises, e.g., from saturable absorption), the term with μ represents, if negative, the saturation of the nonlinear gain, while the one with ν corresponds, also if negative, to the saturation of the nonlinear refractive index. In the rest of the paper we shall always assume $D=1$. Our numerical simulations for solving Eq. (1) are based on the split-step technique using a fast Fourier transform routine for the linear part of the equation.

Equation (1) has a variety of localized solutions. These are stationary solitons, sources, sinks, moving solitons and fronts with fixed velocity [12,13]. A multiplicity of solutions can exist simultaneously. For example, solitons can exist in several forms and many of them can be stable for a certain range of values of the equation parameters [14]. In addition to localized solutions with fixed shape, there are pulsating solitons [8], whose profile changes periodically with propagation distance z. Another interesting discovery is the "exploding soliton" [9]. This localized solution belongs to the class of chaotic solutions. It has intervals of almost stationary propagation, but, over and over again, an instability develops, producing explosions, and then the stationary shape is subsequently recovered.

Another example of an exploding soliton is shown in Fig. 1. It has the same main properties as those explosions we presented in previous works. That is, (1) explosions occur intermittently. In our continuous model, they occur more or less regularly, but the period changes dramatically with a change of parameters. (2) The explosions have similar features, but are not identical. (3) Explosions happen spontaneously, but additional perturbations can trigger them. (4) One of the basic features of this solution is that the recurrence is back to the stationary soliton solution. These characteristics have been observed both theoretically [8,9] and experimentally [10]. In contrast to our previous cases, one of the distinctive features of the present example is that each explosion occurs predominantly on one side of the soliton.

We note here that the explosions observed in the experiment [10] were asymmetric. However, the asymmetry in that case is related to the asymmetry in the system. Explosions

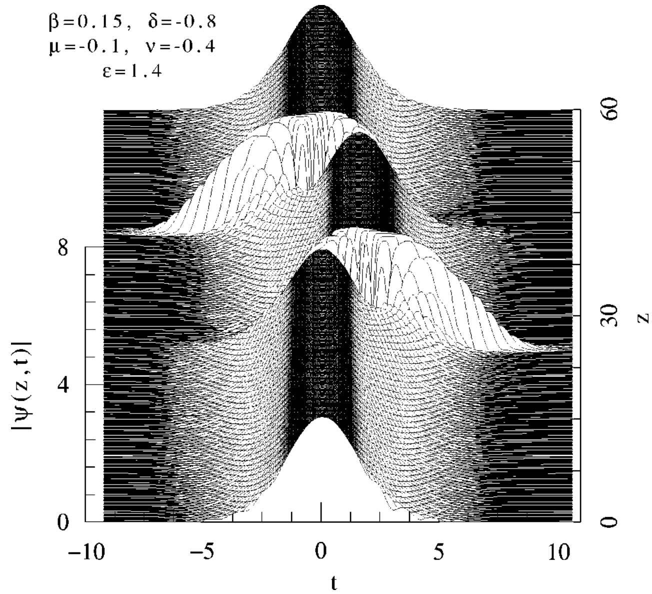

FIG. 1. Soliton evolution showing two consecutive strongly asymmetric explosions. The values of the parameters are given at the top of the figure.

can be made asymmetric if, for instance, we introduce a term in the equation to account for the third-order dispersion in the system. However, in the present case, third-order dispersion is absent and the equation is symmetric relative to a sign change in the t variable. Due to this symmetry, the chances that the explosion will occur at any particular side are equal. In the example shown in Fig. 1 the side on which the explosion occurs changes at each consecutive explosion.

Most of the known soliton solutions are even functions of t, but for small regions of the parameter space, noneven solutions can also be found [14,15]. Explosions, being chaotic solutions in both space and time [9], are asymmetric in the sense that each wing of the pulse breaks into different pieces in a random way. Using the term "symmetric explosion," we mean that both wings of the pulse explode and recover almost simultaneously, in such a way that the center of the pulse remains almost fixed. What we describe here is that for a small region inside the parameter space where explosions occur, these are strongly asymmetric, i.e., both wings explode asynchronously, and after each explosion and subsequent recovery, the center of the solution changes its position.

Because of the asymmetry, the maximum of the soliton in the laminar regime of propagation shifts in z alternately to the left- and right-hand sides after every consecutive explosion. These shifts, for a number of explosions, are shown in Fig. 2. The "vertical" lines in this figure indicate that the peak amplitude is delocalized at the very zenith of the explosion. The initial condition in this particular case is the unperturbed (symmetric) soliton solution for the parameters written inside the figure. The value of the energy Q, defined by $Q(z)=\int_{-\infty}^{\infty}|\psi(t,z)|^2dt$, versus the propagation distance for the same simulation is shown in Fig. 3. After a certain propagation distance where the solution hardly changes, the instability develops from numerical noise producing, first, four big

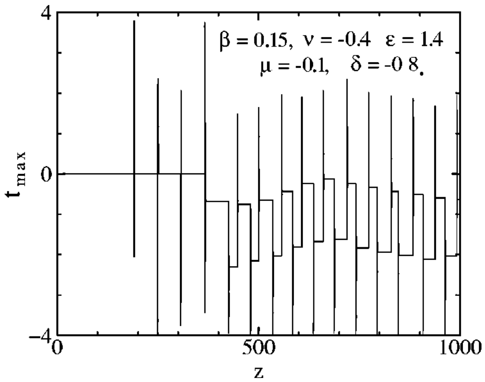

FIG. 2. Temporal position of the field maximum t_{max} versus z. The initial input is the unperturbed soliton solution. The first three explosions are "symmetric." Strongly asymmetric explosions occur after $z=400$.

"symmetric" explosions. These appear only during the transition period. The first four explosions take place on both wings of the pulse and therefore, at its zenith, have almost twice the amount of energy of the asymmetric explosions. Due to the "symmetry," the center of the soliton is not shifted, as Fig. 2 shows. All explosions afterward are strongly asymmetric (two of them are shown in Fig. 1). The separation between consecutive explosions shortens and becomes approximately half of the separation between the first four.

Exploding solitons are intrinsic modes of the system in the same way as stationary solitons. The solution converges to this type of evolution starting from a wide range of initial conditions. In this sense, the soliton with strongly asymmetric explosions serves as an attractor. Figures 2 and 3 show this convergence from a particular initial condition. Similar behavior is observed when we change the equation parameters in certain limits or use another initial shape.

Explosions are one of the many types of instabilities that a soliton can suffer in dissipative systems. Understanding the reasons for such instabilities is important in applications. In fact, a better understanding of instabilities will help to design stable laser configurations. In the case of exploding solitons, the natural approach for studies seems to be linear stability analysis of the soliton in its laminar regime. We note that the soliton is not completely stationary, even in this part of the evolution. However, there is a stationary solution of the CGLE that serves as a "ground state" (partially attracting state) for recoveries after each explosion. The solution approaches this stationary solution with relatively high accuracy before the next instability develops. This stationary solution can be found using, for example, a shooting technique based on z-independent ordinary differential equations [13], chap. 13. The next step is the linear stability analysis of this stationary solution.

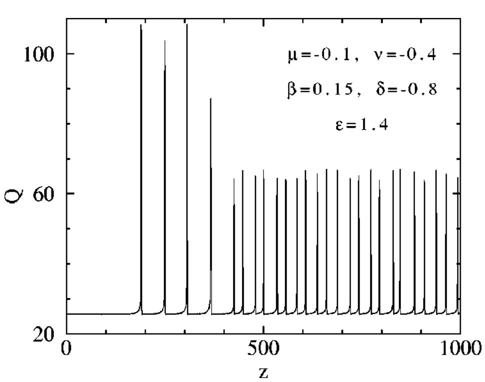

FIG. 3. Energy Q carried by the solution as it propagates.

III. LINEAR STABILITY ANALYSIS OF THE STATIONARY SOLITON

The stationary soliton solution of the CGLE can be written in the form $\psi(z,t)=\psi_0(t)e^{iqz}$, where $\psi_0(t)$ is a complex function of t with exponentially decaying tails. The real number q is its propagation constant. The function $\psi_0(t)$ can be calculated numerically when the analytic form is unknown. The stationary soliton solution is a singular point of this dynamical system in an infinite-dimensional phase space. The evolution of the solution in the vicinity of this singular point can be described by

$$\psi(z,t) = [\psi_0(t) + f(t)e^{\lambda z} + h(t)e^{\lambda^* z}]e^{iqz}, \tag{2}$$

where $f(t)$ and $h(t)$ are small perturbation functions (we assume $|f,h| \ll |\psi_0|$ for any t), and λ is the associated perturbation growth rate. In general, all λ's are complex numbers and f and h are complex functions. Substituting Eq. (2) into the CGLE Eq. (1), we obtain

$$\begin{aligned}(i\lambda - i\delta - q)fe^{\lambda z} &+ (i\lambda^* - i\delta - q)he^{\lambda^* z} + (D/2 - i\beta)f_{tt}e^{\lambda z} \\ &+ (D/2 - i\beta)h_{tt}e^{\lambda^* z} + 3(\nu - i\mu)|\psi_0|^4(fe^{\lambda z} + he^{\lambda^* z}) \\ &+ 2(\nu - i\mu)|\psi_0|^2\psi_0^2(f^* e^{\lambda^* z} + h^* e^{\lambda z}) + 2(1 - i\epsilon) \\ &\times|\psi_0|^2(fe^{\lambda z} + he^{\lambda^* z}) + (1 - i\epsilon)\psi_0^2(f^* e^{\lambda^* z} + h^* e^{\lambda z}) = 0.\end{aligned} \tag{3}$$

Separating terms with different functional dependencies on z, we obtain the following two coupled ordinary differential equations:

$$\begin{aligned}Af + Bf_{tt} + Ch^* &= \lambda f, \\ A^* h^* + B^* h_{tt}^* + C^* f &= \lambda h^*,\end{aligned} \tag{4}$$

where,

$$A = \delta - iq + 2(\epsilon + i)|\psi_0|^2 + 3(\mu + i\nu)|\psi_0|^4,$$

$$B = \beta + i\frac{D}{2},$$

$$C = [\epsilon + i + 2(\mu + i\nu)|\psi_0|^2]\psi_0^2.$$

The technique for solving Eqs. (4) numerically has been described in [11]. Here, we use the following parameters for the CGLE: $\mu=-0.1$, $\epsilon=1.4$, $\beta=0.15$, and $\delta=-0.8$, while $\nu=-0.4$. The complex plane, with the eigenvalues obtained as described in [11], is shown in Fig. 4. The total spectrum

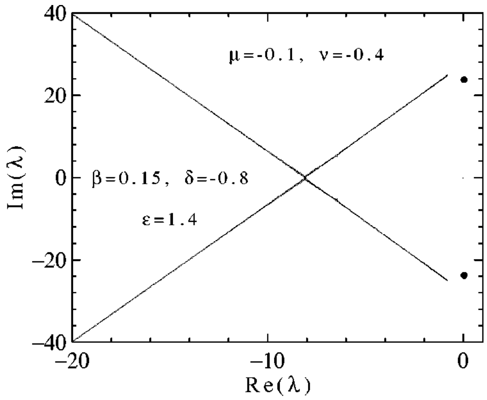

FIG. 4. The spectrum of eigenvalues in the complex plane for an exploding soliton.

consists of two complex conjugate eigenvalues with positive real parts and a continuous spectrum of complex conjugate eigenvalues, all with negative real parts. An important observation is that the two complex conjugate eigenvalues with positive real parts are almost degenerate. There are two eigenfunctions corresponding to almost the same eigenvalue, where one is an even function of t and the other is odd. The real part of the eigenvalue, when positive, is usually called the growth rate of the perturbation, and we will henceforth denote it as g.

These two eigenvalues with the highest growth rates can also be found using an alternative technique [16] by imposing the desired symmetry through the boundary conditions. We used both methods. Figure 5(a) shows the largest growth rate of the even and odd perturbations associated with the stationary solution corresponding to $\mu=-0.1$, $\epsilon=1.4$, $\beta=0.15$, and $\delta=-0.8$, vs ν. The two curves are indistinguishable on the scale of this figure. The difference between them is shown in Fig. 5(b). The growth rate for the odd perturbation function is always slightly greater. The difference diminishes as ν increases ($|\nu|$ decreases) and the growth rate itself increases considerably, so that the relative difference quickly tends to zero. On the other hand, the difference increases when the growth rate tends to zero and the stationary solution tends to be stable. Although the figure concentrates on what happens when we change ν, we have obtained similar results when changing any of the parameters around the point $(\delta,\beta,\epsilon,\nu,\mu)=(-0.8,0.15,1.4,-0.4,-0.1)$. The g dependence on variable β is shown in Fig. 6. In this section of the parameter space, the real part of the eigenvalue becomes zero at $\beta\approx 0.162$. Considering all such plots, we can find a surface in the five-dimensional parameter space where the real part of the eigenvalues is zero (the surface of neutral stability).

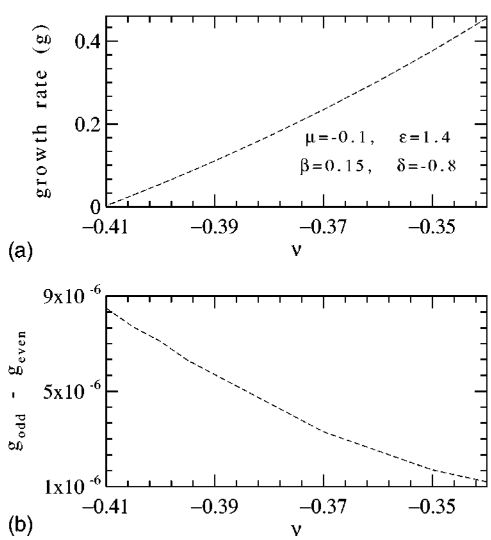

FIG. 5. (a) Largest growth rate for even (dashed line) and odd (dotted line) modes of perturbation as a function of ν. The two lines are indistinguishable on the scale of the figure. (b) Difference between the largest growth rates for odd and even mode perturbation.

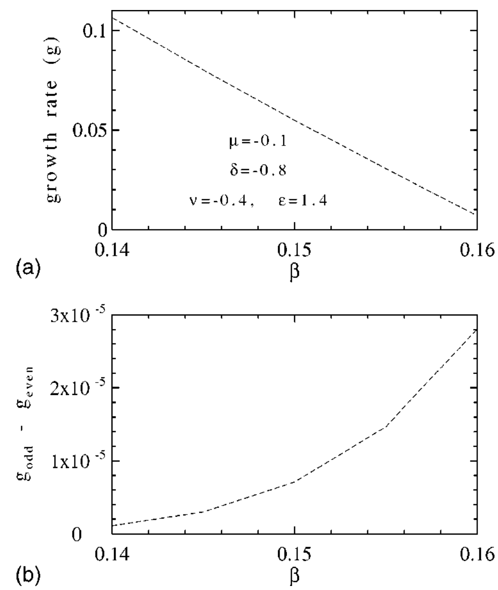

FIG. 6. (a) Largest growth rate for even (dashed line) and odd (dotted line) modes of perturbation as a function of β. (b) Difference between the largest growth rate for odd and even modes of perturbation.

Figure 7 illustrates the profiles of the (a) even and (b) odd eigenfunctions (f,h) with the largest growth rates. With the solid line, we also show the corresponding stationary solution with which they are associated. The distinctive feature of these perturbation functions is that they consist of two well-separated parts. At the top of the soliton, the perturbation function is close to zero and becomes nonzero on the soliton wings. This property, together with the slightly different eigenvalues, is the key for understanding the phenomenon of strongly asymmetric explosions.

IV. GLOBAL DYNAMICS OF EXPLODING SOLITONS

A. Explosion as a homoclinic orbit

Before explaining the reasons for strongly asymmetric explosions, let us consider the global dynamics of the unstable soliton after the linear stage of instability. In the presence of eigenvalues with positive real parts, the soliton evolution undergoes the following transformation. Suppose, initially, we have the stationary solution with small perturbations. We

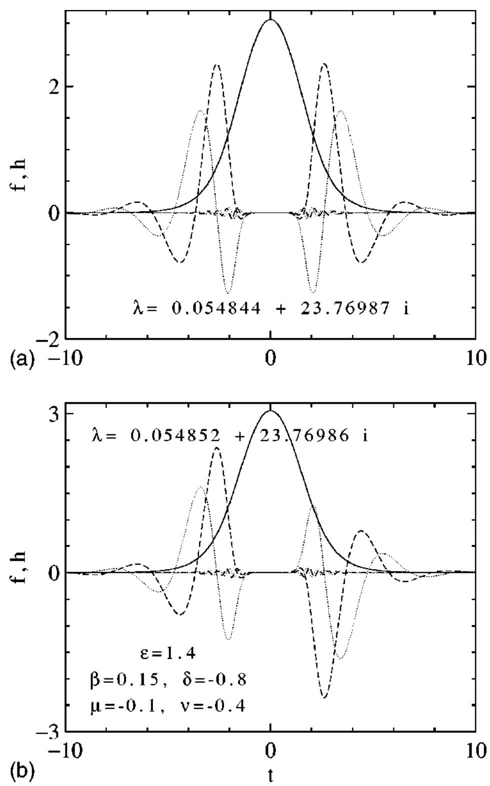

FIG. 7. Real (dotted line) and imaginary (dashed line) parts of (a) the even and (b) the odd perturbation functions, normalized for convenience. The solid lines in (a) and (b) show the amplitude of the soliton itself.

note that the real parts of the eigenvalues are relatively small, so that perturbations grow slowly. The imaginary parts of the eigenvalues (which are usually quite large) result in (fast) oscillations whose amplitude increases on propagation. We also note that the soliton center is not influenced by this instability, because the eigenfunctions are almost zero in the central part of the soliton. Thus, these fast oscillations, with exponentially growing amplitudes, initially occur only in the wings of the solution.

After the initial linear growth of the perturbation, its amplitude becomes comparable with the soliton amplitude, and the dynamics becomes strongly nonlinear. The nonlinearity mixes all perturbations, creating radiative waves. The amplitudes of radiative waves increase at the expense of the initial perturbation. Consequently, the fraction of the initial perturbation within them becomes small. The solution at this stage appears to be completely chaotic. However, the solution remains localized, both in amplitude and in width, due to the choice of the system parameters. In particular, the maximum field amplitude is limited, due to the fact that μ is negative. In addition, a positive β ensures that the total width in the frequency domain also stays finite, provided that other parameters are within certain ranges. It is also important that the stationary soliton shape is fixed, thus providing the point of return. As all radiative waves have eigenvalues with negative real parts, they decay and quickly disappear, since the eigenvalues for most of them have negative real parts with a larger modulus than the initial perturbation. This means that

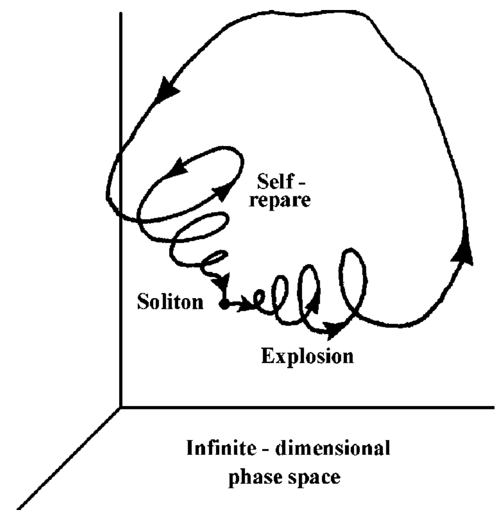

FIG. 8. One cycle of evolution of an exploding soliton in an infinite-dimensional phase space.

the solution returns to the state of a stationary soliton with a small perturbation that has an eigenvalue with positive real part. As the real part of the discrete eigenvalue is relatively small, the instability develops again later, thus repeating the whole period of the evolution described above. This process is repeated indefinitely along the z axis. One cycle of this evolution is shown, schematically, in Fig. 8.

The fixed point, shown by a black dot in this figure, corresponds to the stationary soliton solution. It can be classified as a stable-unstable focus, because all the eigenvalues in the stability analysis appear as complex conjugate pairs. We stress here that our system has an infinite number of degrees of freedom, and that the evolution actually occurs in an infinite-dimensional phase space. It cannot be reduced to a finite-dimensional problem, as all the eigenvalues play essential roles in the dynamics. As the fixed point is unstable, the trajectory leaves it in the direction in the phase space defined by the discrete eigenvalues. This motion is exponential as well as oscillatory. After complicated dynamics in the whole phase space, the trajectory, being homoclinic, returns to the same fixed point but along a different route, as defined by the continuous spectrum. This return is also accompanied by oscillations, as all the eigenvalues in this problem are complex. This scenario is similar to the one described by Shil'nikov's theorem [11,17].

This process can also be compared to the relaxation oscillations in nonlinear systems with one degree of freedom [18]. The frequency of those oscillations is related to the relaxation parameters of the system rather than to real frequencies. Our exploding soliton is an example of a relaxation oscillation in a system with an infinite number of degrees of freedom. The main feature of these oscillations is that all complex eigenvalues play an important role in these oscillations. As with any other relaxation oscillations, the period of the oscillations is not fixed and varies from one explosion to another.

B. Perturbation functions as supermodes of a coupler

The soliton instability becomes more complicated when there are several modes of perturbation with positive real parts. As we have only two modes, the changes of the soliton shape can still be explained using a relatively simple approach. For a soliton, the perturbation function is essentially a mode of a waveguide. The splitting of the function into two parts, as shown in Fig. 7, makes it look more like a mode of a coupler rather than a mode of a single waveguide. We know that a coupler can have so-called supermodes. These are symmetric and antisymmetric combinations of modes of each of the waveguides comprising the coupler. We also know that these supermodes have slightly different propagation constants.

Similarly, our symmetric and antisymmetric perturbation functions have slightly different eigenvalues. The difference between the eigenvalues is defined by the distance between the left- and right-hand sides of the perturbation functions, which are separated by a region of exponentially decaying amplitudes from each side. As a consequence this difference is exponentially small. We have been able to distinguish the difference between the eigenvalues numerically [see, e.g., Fig. 5(b)]. Nonetheless, even though small, this difference can play an important role in the dynamics.

A combination of symmetric and antisymmetric modes produces an asymmetric perturbation. Hence, generally speaking, the explosions are never symmetric. The chaotic soliton profile during the explosions is neither an even nor an odd function of t. It comes close to being an even function only during the laminar regime of evolution. The almost equal values of the eigenvalues for even and odd perturbation functions make the explosions asymmetric, in the sense that each explosion occurs predominantly on one side of the soliton. Then, the growth of the perturbation at one wing of the soliton can be suppressed, and at the opposite wing can be enhanced. We observe this interesting feature clearly in Fig. 1.

V. ENERGY CONSIDERATIONS

A. Energy flow across the soliton

The above results show that the soliton is stable in its central part but unstable in its wings. To some extent, this feature can be understood if we recall that solitons in dissipative systems have parts generating energy and parts dissipating it. The analysis can be done, based on the continuity relation for the CGLE equation, and this can be written in the form

$$\frac{\partial \rho}{\partial z} + \frac{\partial j}{\partial t} = P, \tag{5}$$

where ρ is the energy density,

$$\rho = |\psi|^2.$$

The corresponding flux j is

$$j = \frac{iD}{2}(\psi\psi_t^* - \psi_t\psi^*),$$

and the density of energy generation P is

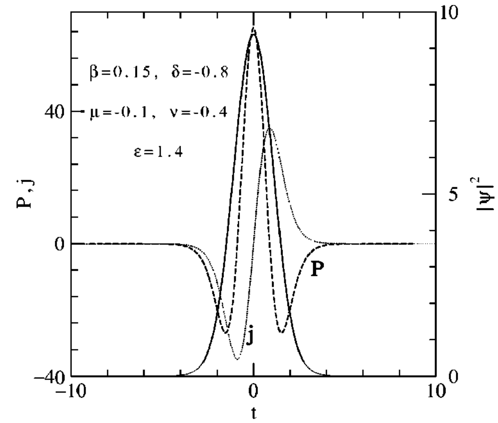

FIG. 9. Energy generation $P(t)$ (dashed line) and energy flux $j(t)$ (dotted line) across the soliton. The soliton profile $|\psi|$, denoted by the solid line, is included for comparison (RHS scale).

$$P = 2\delta|\psi|^2 + 2\epsilon|\psi|^4 + 2\mu|\psi|^6 - 2\beta|\psi_t|^2 + \beta(|\psi|^2)_{tt}.$$

The last term distinguishes Eq. (5) from the conservative system where P is zero. For a stationary solution, the energy density $|\psi|^2$ does not depend on z. Therefore, the first term in Eq. (5) is zero. Hence, the energy flux j is defined by the regions of energy generation and loss. The energy flows from the parts where the energy is generated to the parts where it is dissipated. A large energy flow may create turbulence and corresponding instability.

Figure 9 shows the curve for the energy flux j across the soliton as a dotted line. It is zero in the center of the soliton and has opposite signs at its wings, reflecting the fact that energy flows from the center to the tails. In the figure, we also show $P(t)$ as a dashed line and the soliton itself by a solid line.

B. Three stages of the soliton cooling process

The central part of the soliton is the source of the energy generation, as we can see from the curve for P in Fig. 9. Literally speaking, it is the hottest part of the soliton. The energy flows from the center to the sides of the soliton and is dissipated at its wings. This process is nothing other than soliton cooling. Depending on the parameter values of the system, soliton cooling can be laminar, wave assisted, or turbulent. When the soliton is stable, the shape is a smooth function of t and the energy flow is laminar. This happens at values of ν below −0.41 when the real part of the eigenvalues is negative.

Keeping the rest of the parameters fixed, at values of ν around −0.41, the discrete eigenvalues obtained from the linear stability analysis described in Sec. III cross the vertical axis in the complex plane, thus gaining a positive real part. As soon as the real part becomes positive, the instability develops. If this growth rate is small enough, any perturbation will develop until saturation is reached, giving rise to periodic wave motions in the soliton wings. In this regime, the energy flow to the soliton tails exceeds the value at which laminar cooling is possible. Small amplitude waves appear in the soliton wings and these assist the soliton cooling process.

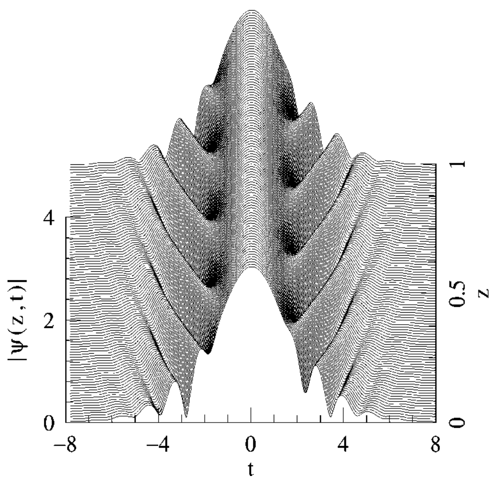

FIG. 10. Antisymmetric oscillations around the soliton.

Figure 10 shows an example, for $\nu=-0.41$, of such solitons with waves in both wings. These waves are defined by the two sets of perturbation functions, and the frequency of the oscillations is very close to the imaginary part of the eigenvalue with the largest real part. The closer we are to the bifurcation point, the closer is its angular frequency to this imaginary part. The symmetric and antisymmetric perturbation functions can be involved in various proportions, depending on the initial conditions. As a result, the symmetry of the oscillations is determined by the initial conditions. Figure 10 shows antisymmetric oscillations around the soliton. The oscillations can also be symmetric, as shown in Fig. 11, or any combination of the two, in accordance with the fact that the soliton has even and odd perturbation modes.

The soliton cooling process is still laminar but wave assisted here. The periodicity in the z direction, related to waves in the wings, can be seen clearly from Fig. 12, which shows the soliton intensity at a fixed t of the soliton profile, namely, at ± 3 from the center of the soliton, versus z. Similar periodic behavior occurs if we plot the soliton intensity for any other value of t.

Different modes of oscillation have different phase delays between oscillations in the right or left wings of the soliton.

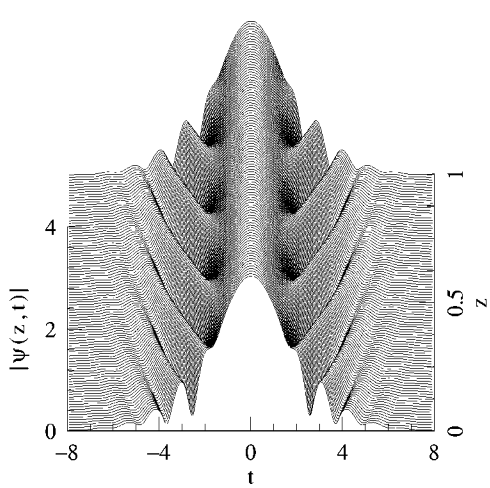

FIG. 11. Symmetric oscillations around the soliton.

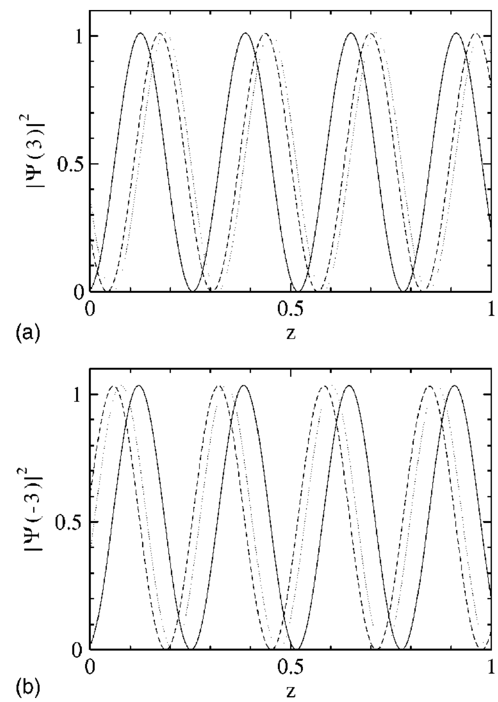

FIG. 12. Soliton intensity $|\psi|^2$ versus z for the same equation parameters but for three different initial conditions. The curves in (a) are for the rhs tail (namely, for $t=+3$) and the curves in (b) are for the LHS tail ($t=-3$). The oscillations are independent and can be in phase (solid line), out of phase (dotted line), or at another, different phase difference (dashed line).

The presence of even and odd perturbation functions creates a situation where the two wings of the soliton seem to oscillate independently. The three curves in Fig. 12 are for the same equation parameters but different initial conditions. Each initial condition inevitably converts, after some propagation, into a periodic solution with fixed amplitude and period, but with a random phase relationship between the oscillations of each wing.

On further increase of the parameter ν, the wave amplitude at the soliton wings increases, providing higher energy flow and better cooling conditions. At certain ν, the wave-assisted stage also becomes impossible and waves are transformed into chaotic formations, thus, transforming the stationary soliton into an exploding one. This consideration allows us to distinguish three clearly pronounced stages of the soliton cooling process.

We can formally compare the three stages of the soliton cooling process with the processes occurring in Rayleigh-Bénard convection experiments [19]; namely, the Rayleigh-Bénard phenomenon is an instability of a fluid which is confined in a thin layer, and is heated from below to produce a fixed temperature difference. Depending on the temperature difference, the process of heat transfer changes. When it is low, we have homogeneous heat conduction, at higher values, the convection pattern appears, and if the temperature difference is very large, then the fluid rises very quickly, and a turbulent flow may be created. We should stress that at this stage we can only give this analogy rather than compare these two phenomena in detail.

VI. DISCUSSION

Nonlinear dynamics operates with phase space and special points and objects in this space. We can link our results to these objects developed in the case of systems with a finite number of degrees of freedom. There is never a one-to-one correspondence, but common features between these objects can certainly be noticed. First of all, the stationary soliton is fixed in z, and therefore it is a singular point of a system, just like any other stationary solution. Pulsating solitons change their shapes periodically, and therefore they are equivalent to limit cycles. They all have the basic properties of limit cycles. They bifurcate from stationary solitons at certain values of parameters. The periodic motion becomes stable after this bifurcation, while the stationary soliton becomes unstable. Nearby trajectories in the phase space converge to the pulsating soliton, rather than to the stationary solutions. This convergence occurs in the same way as trajectories in two-dimensional phase space converge to stable limit cycles.

Exploding solitons have a counterpart in finite-dimensional dynamics in the form of relaxation oscillations [18]. We recall that relaxation oscillations also form a limit cycle which is more complicated than a simple circular structure. The period of relaxation oscillations is related to relaxation parameters in the system, rather than to real frequencies. At the same time, when the number of degrees of freedom increases, the dynamics becomes more complicated. Our present study shows these complications. It shows that, in addition to the simple, almost periodic behavior in z, the soliton can have lateral instabilities with energy bursts shifting to the left or right in each explosion.

Alternatively, exploding solitons can be considered as strange attractors. The exploding soliton solution is chaotic rather than regular. At the same time, the solution stays localized and it does not leave a certain region in the phase space. Moreover, nearby trajectories converge to this region, so that a localized initial condition becomes an exploding soliton after some distance of propagation. These analogies help us to understand the phenomenon to some extent, but careful studies using linear stability analysis and beyond are needed to give a more complete picture.

VII. CONCLUSIONS

In conclusion, we have found that, at certain values of the parameters, solitons can have strongly asymmetric explosions. These have a counterpart in nonlinear dynamics with a finite number of degrees of freedom in the form of relaxation oscillations. We gave qualitative explanations for this effect, after carrying out a linear stability analysis. We revealed a three-stage soliton cooling process in the transition from stationary to exploding solitons. These results can be helpful in predicting instabilities of passively mode-locked lasers and, as a result, in helping to make them more stable.

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